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Incompressible viscous flow past a semi-infinite flat plate

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The asymptotic solution for the incompressible viscous flow past a semi-infinite flat plate constructed by Goldstein (1956, 1960) can be valid only if the solutions of certain ordinary differential equations obey certain constraints (given in Goldstein 1956, 1960). In this paper, we construct the solutions of these equations, show that the necessary constraints are met, and hence establish the validity of the asymptotic solution up to the order considered. The manner in which undetermined constants appear in the solution are discussed.

Introduction

Goldstein (1956, 1960) obtains an asymptotic description of the stream function ψ_1, for the flow, with velocity U, of an incompressible viscous fluid, with viscosity μ, past a semi-infinite flat plate which in Cartesian co-ordinates lies in $x_1 > 0$, $y_1 = 0$. With

$$\psi = \psi_1/\mu, \quad \zeta' = \zeta e^{-\frac{1}{2}\pi i}, \quad \zeta = \xi + i\eta = (U/\nu)^{\frac{1}{2}}(x_1 + iy_1)^{\frac{1}{2}}, \tag{1}$$

where ν is the kinematic viscosity and ξ, η are the conventional parabolic co-ordinates stretched by $(U/\nu)^{\frac{1}{2}}$. The plate lies in $\eta = 0$ and the flow field occupies the region $-\infty < \xi < \infty$, $\eta > 0$. The least general potential solution into which the boundary-layer solution merges is given by (see Goldstein 1960) $\psi = \operatorname{Im} w$, where w is given by

$$w = -\zeta'^2 + \beta\zeta' + \sum_{m=1} \frac{1}{\zeta'^m}[b_{m,m}(\log\zeta')^m + b_{m,m-1}(\log\zeta')^{m-1} + \ldots + b_{m,1}\log\zeta' + b_{m,0}], \tag{2}$$

where $\beta = 1{\cdot}7208$, and the $b_{i,j}$ are real. The expansion of w for large ξ ($|\xi| > \eta$) suggests that the appropriate form for the asymptotic solution for ψ in the boundary layer is

$$\begin{aligned}\psi = {} & \xi f_0(\eta) + \xi^{-1}[f_2(\eta) + g_2(\eta)\log\xi] \\ & + \xi^{-2}[f_3(\eta) + g_3(\eta)\log\xi + h_3(\eta)(\log\xi)^2] \\ & + \xi^{-3}[f_4(\eta) + g_4(\eta)\log\xi + h_4(\eta)(\log\xi)^2 + k_4(\eta)(\log\xi)^4] + \ldots,\end{aligned} \tag{3}$$

where the functions $f_n, g_n, h_n, k_n, \ldots$ and their first derivatives vanish at the origin and, by comparison with (2) for ξ large and $|\xi| > \eta$,† must be such that

$$\left.\begin{aligned}
&f_0 \sim 2\eta-\beta, \quad f_2 \sim b_{10}, \quad g_2 \sim b_{11},\\
&f_3 \sim \tfrac{1}{2}\pi(b_{21}-b_{11}\eta), \quad g_3 \sim \pi b_{22}, \quad h_3 \sim 0,\\
&f_4 \sim (\tfrac{1}{4}\pi^2 b_{32}-b_{30})-\eta(\tfrac{1}{2}\pi^2 b_{22}+b_{21}-2b_{20})+\eta^2(\tfrac{3}{2}b_{11}-b_{10}),\\
&g_4 \sim (\tfrac{3}{4}\pi^2 b_{33}-b_{31})-2\eta(b_{22}-b_{21})-b_{11}\eta^2,\\
&h_4 \sim -b_{32}+2b_{22}\eta, \quad k_4 \sim -b_{33},
\end{aligned}\right\} \tag{4}$$

the error terms being exponentially small. Goldstein (1960) gives a complete discussion of the form of the solution: the discussion rests on the conjecture that certain ordinary differential equations admit solutions with specifically restricted asymptotic behaviour. Imai (1957), in a completely independent investigation, used exactly the same form for the $1/\zeta'$, $(\log\zeta')/\zeta'$, $1/\xi$ and $(\log\xi)/\xi$ terms and his numerical calculation for the constant b_{11} (see equations (3), (4)) is in agreement with that found below.

2. Differential equations and their solutions

Substitution of (3) in the full Navier–Stokes equation, in appropriate parabolic co-ordinate‡ form, shows that the f, g, h, k, must obey the following differential equations:§

$$\left.\begin{aligned}
&f_0''' + f_0 f_0'' = 0,\\
&L_2(g_2) = 0, \quad L_2(f_2) = F_2,\\
&L_3(h_3) = 0, \quad L_3(g_3) = F_{31}, \quad L_3(f_3) = F_{32},\\
&L_4(k_4) = 0, \quad L_4(h_4) = F_{41}, \quad L_4(g_4) = F_{42}, \quad L_4(f_4) = F_{43},
\end{aligned}\right\} \tag{5}$$

where f_0 is the Blasius function, $f_0 \sim 2\eta-\beta$, $\beta = 1{\cdot}7208$, $f_0''(0) = \alpha = 1{\cdot}32824$, $f_0'' \sim \gamma\exp(-\lambda^2)$, where $\lambda = \eta - \frac{1}{2}\beta$, and γ is a constant, the primes denote differentiation with respect to η, and

$$\left.\begin{aligned}
F_2 &= d(\eta f_0'-f_0)^2/d\eta + f_0' g_2'' - f_0''' g_2,\\
F_{31} &= 2[h_3'' f_0' - h_3 f_0'''], \quad F_{32} = f_0' g_3'' - f_0''' g_3,\\
F_{41} &= g_2 g_2''' - 3g_2' g_2'' + 3[f_0' k_4'' - f_0''' k_4],\\
F_{42} &= [-g_2 g_2''' + g_2' g_2'' - 2g_2' f_0 - 10 g_2 f_0' - 12 g_2'']\\
&\quad + 2\eta[2g_2''' + f_0 g_2'' - f_0'' g_2]\\
&\quad + 2\eta^2[f_0' g_2'' + f_0'' g_2'] + 2[f_0' h_4'' - f_0''' h_4]\\
&\quad + [g_2 f_2''' - 3g_2' f_2'' - 3g_2'' f_2' + g_2''' f_2],\\
F_{43} &= [3f_0 g_2' + 10 g_2'' + 17 f_0' g_2 + 2\eta f_0'' g_2 - 2\eta^3 f_0''(\eta f_0' - f_0)]\\
&\quad + [-(g_2 f_2''' + 12 f_2'' + 2 f_0 f_2' - g_2'' f_2' + 10 f_0' f_2)\\
&\quad + 2\eta(2f_2''' + f_0 f_2'' - f_0'' f_2)\\
&\quad + 2\eta^2(f_0' f_2'' + f_0'' f_2') + (f_0' g_4'' - f_0''' g_4)]\\
&\quad + [f_2 f_2''' - 3 f_2' f_2''].
\end{aligned}\right\} \tag{6}$$

† Since we are considering a descending series in ξ, $\tan^{-1}\xi/\eta$ must be replaced by $\frac{1}{2}\pi - \tan^{-1}\eta/\xi$ for $\xi > 0$ and $-\frac{1}{2}\pi - \tan^{-1}\eta/\xi$ for $\xi < 0$.

‡ These are the optimal co-ordinates (see Kaplun 1954).

§ The operator $L_n = \dfrac{d^4}{d\eta^4} + f_0\dfrac{d^3}{d\eta^3} + (n+1)f_0'\dfrac{d^2}{d\eta^2} + f_0''\dfrac{d}{d\eta} - (n-1)f_0'''$.

It is shown in Goldstein (1960) following Whittaker & Watson (1932) that the two complementary functions of $L_n(y) = 0$ with double zeros at the origin, denoted by $y_n^{(2)}$, $y_n^{(3)}$, are

$$\left.\begin{aligned} y_n^{(2)} &= \alpha\eta^2/2! - (n+2)\,\alpha^2\eta^5/5! + \ldots, \\ y_n^{(3)} &= \alpha\eta^3/3! - 2(n+2)\,\alpha^2\eta^6/6! + \ldots, \end{aligned}\right\} \tag{7}$$

for small η, and for large η,

$$\left.\begin{aligned} y_n^{(2)} &\sim a_n^{(2)} + b_n^{(2)} E_n + c_n^{(2)} H_n, \\ y_n^{(3)} &\sim \alpha\eta/2n + a_n^{(3)} + b_n^{(3)} E_n + c_n^{(3)} H_n, \end{aligned}\right\} \tag{8}$$

where
$$\left.\begin{aligned} E_n &\sim \lambda^{1-n}[1 + (n-1)\,n/4\lambda^2 + \ldots], \\ H_n &= \exp(-\lambda^2)\,\lambda^{n-2}[1 - (n-2)(n-3)/4\lambda^2 + \ldots], \end{aligned}\right\} \tag{9}$$

and $a_n^{(2)}$, $b_n^{(2)}$, $c_n^{(2)}$, $a_n^{(3)}$, $b_n^{(3)}$, $c_n^{(3)}$ are constants. For each n, the $y_n^{(2)}$ and $y_n^{(3)}$ (except for $y_2^{(2)}$) were calculated numerically by solving $L_n(y_n) = 0$ in steps of $\eta = 0{\cdot}02$ with initial boundary conditions

$$\left.\begin{aligned} y_n^{(2)} &= 0 = y_n^{(2)\prime} = y_n^{(2)\prime\prime}, \quad y_n^{(2)\prime\prime\prime} = \alpha, \\ y_n^{(3)} &= 0 = y_n^{(3)\prime} = y_n^{(3)\prime\prime\prime}, \quad y_n^{(3)\prime\prime} = \alpha, \end{aligned}\right\} \quad \eta = 0.$$

After computing the E_n, the numerical solutions so found were then equated to the asymptotic form (9), at a value of η where the exponential terms were negligible, and the constants $a_n^{(2)}$, $a_n^{(3)}$, $b_n^{(2)}$, $b_n^{(3)}$ evaluated. It was found that none of these $a_n^{(i)}$, $b_n^{(i)}$ $(i = 2, 3)$ were zero for n up to 4, except $b_2^{(2)}$, which case is discussed below. The particular integrals were found in a similar way and none of $a_n^{(i)}$, $b_n^{(i)}$ $(i = 4, \ldots, 9)$ defined below and in the appendix were found to be zero.

The boundary conditions on the f_n, g_n, h_n, k_n are that they must have double zeros at the origin and be asymptotic to the values in (4) with *exponentially* small errors. Since equations (5) are linear the f_n, g_n, h_n, k_n will involve linear combinations of the complementary functions and particular integrals, and must be such that the coefficient of the E_n in each of them must be zero (see Goldstein 1960). In view of the complicated form of the F_n in (6) each function must be treated separately to ensure that this is possible. It is necessary, as shown below and in the appendix, to consider the functions in the order, k, h, g, f for each n. The case $n = 2$ is a special case and will be discussed in detail. The case $n = 3$ will also be discussed as a more typical case. All other cases are treated comparably. The complication increases with n, and the case $n = 4$ is given for reference in the appendix.

Case $n = 2$. The equation for $g_2(\eta)$ is, from (5),

$$L_2(g_2) = 0.$$

This equation is an exception to the general form since

$$y_2^{(2)} = \eta f_0' - f_0,$$

and so $b_2^{(2)} \equiv 0$. Thus with (4)

$$g_2(\eta) = b_{11}(\eta f_0' - f_0)/\beta, \tag{10}$$

where b_{11} is as yet undetermined.

The equation for $f_2(\eta)$ is given by (5) and (6). Complementary functions are (since $b_2^{(2)} \equiv 0$)

$$y_2^{(2)} \sim a_2^{(2)} = \beta, \quad y_2^{(3)} \sim \tfrac{1}{4}\alpha\eta + a_2^{(3)} + b_2^{(3)} E_2 + c_2^{(3)} H_2.$$

Let $y_2^{(4)}$ be the particular integral of the f_2 equation with the first term in F_2 only and $y_2^{(5)}$ that with the second term in F_2 only and $b_{11} = \beta$. Thus from (5) and (6),

$$\left.\begin{aligned} y_2^{(4)} &\sim \tfrac{1}{4}\beta^2\eta + a_2^{(4)} + b_2^{(4)} E_2 + c_2^{(4)} H_2, \\ y_2^{(5)} &\sim a_2^{(5)} + b_2^{(5)} E_2 + c_2^{(5)} H_2, \end{aligned}\right\}$$

and so
$$f_2(\eta) = y_2^{(4)} + b_{11} y_2^{(5)}/\beta + A y_2^{(2)} + B y_2^{(3)},$$

where A, B are constants at our disposal. Since, from (4), $f_2 \sim b_{10}$, A, B must be chosen to annul the η and E_2 terms in the above expression for f_2. Thus

$$B = -\beta^2/\alpha, \quad b_{11} = (\beta/\alpha b_2^{(5)})(\beta^2 b_2^{(3)} - \alpha b_2^{(4)}), \tag{11}$$

which thus determines b_{11}. With these values

$$f_2(\eta) \sim b_{10} = [a_2^{(4)} + (a_2^{(5)}/\alpha b_2^{(5)})(\beta^2 b_2^{(3)} - \alpha b_2^{(4)}) - (\beta^2/\alpha) a_2^{(3)}] + A\beta.$$

Thus b_{10} is undetermined since A is undetermined.

Note that the construction of $f_2(\eta)$ depends on $b_2^{(3)}, b_2^{(4)}, b_2^{(5)}$ being non-zero: this was found to be the case numerically. Although the fact that $b_2^{(2)} \equiv 0$ is exceptional, the solution for $f_2(\eta)$ can still be constructed. At this stage *one* undetermined constant is introduced.

Write

$$\left.\begin{aligned} f_2(\eta) &= {}_1f_2(\eta) + A\, {}_2f_2(\eta), \quad b_{10} = {}_1b_{10} + A\, {}_2b_{10}, \\ \text{where} \quad {}_1f_2 &= y_2^{(4)} - \beta^2 y_2^{(3)}/\alpha + b_{11} y_2^{(5)}/\beta, \quad {}_2f_2 = y_2^{(2)} = \eta f_0' - f_0, \\ {}_1b_{10} &= [a_2^{(4)} + (a_2^{(5)}/\alpha b_2^{(5)})(\beta^2 b_2^{(3)} - \alpha b_2^{(4)}) - (\beta^2/\alpha) a_2^{(3)}], \quad {}_2b_{10} = \beta. \end{aligned}\right\} \tag{12}$$

The functions $g_2(\eta), g_2'(\eta), {}_1f_2(\eta), {}_1f_2'(\eta), {}_2f_2(\eta), {}_2f_2'(\eta)$ are tabulated in table 1,† and the second derivatives at $\eta = 0$ in table 5. Also

$${}_1b_{10} = -2{\cdot}2062 \quad \text{and} \quad {}_2b_{10} = \beta = 1{\cdot}7208.$$

If the solution were terminated at this stage an undetermined constant A (or b_{10}) appears. This will allow this solution to be joined onto that valid near the leading edge.

Case $n = 3$. From (5), $h_3(\eta)$ can only be a combination of complementary functions. No linear combination of $y_3^{(2)}$ and $y_3^{(3)}$ is possible which annuls both the η and E_3 terms so that the combination asymptotes to zero with an exponentially small error. Therefore $h_3(\eta) \equiv 0$.

Since $h_3(\eta) \equiv 0$, the equation for $g_3(\eta)$, from (5) and (6), becomes

$$L_3(g_3) = 2(h_3'' f_0' - h_3 f_0''') = 0. \tag{13}$$

By the same argument as for $h_3(\eta)$, $g_3(\eta) \equiv 0$ and $b_{22} = 0$ (equation (4)).

The fact that $h_3(\eta) \equiv 0$ and $g_3(\eta) \equiv 0$ depends on $a_3^{(2)}, b_3^{(2)}$ being non-zero, which was found to be the case numerically.

† Tables 1–4 have been lodged with the Editor and may be borrowed on request.

Since $g_3(\eta) \equiv 0$ the equation for $f_3(\eta)$ from (5) and (6) becomes

$$L_3(f_3) = f_0' g_3'' - f_0''' g_3 = 0.$$

From (4), $f_3(\eta) \sim \frac{1}{2}\pi(b_{21} - b_{11}\eta)$, so it is possible to combine $y_3^{(2)}$ and $y_3^{(3)}$ to annul the E_3 term and asymptote to the above, giving

$$f_3(\eta) = (3\pi b_{11}/\alpha b_3^{(2)})\,(b_3^{(3)} y_3^{(2)} - b_3^{(2)} y_3^{(3)}), \tag{14}$$

and

$$b_{21} = (6b_{11}/\alpha b_3^{(2)})\,(b_3^{(3)} a_3^{(2)} - b_3^{(2)} a_3^{(3)}). \tag{15}$$

Clearly with $a_3^{(2)}$, $b_3^{(2)}$ not equal to zero, as is the case, the construction of $f_3(\eta)$ depends on $b_3^{(3)}$ being non-zero. This was found to be so numerically.

$f_3(\eta)$, $f_3'(\eta)$ are tabulated in table 2,† $f_3''(0)$ in table 5, and $2{\cdot}2670 \leqslant b_{21} \leqslant 2{\cdot}2673$. Note that the $n = 3$ case introduces no more undetermined constants.

The $n = 4$ case is carefully discussed in the appendix and it is shown that a further undetermined constant B is introduced.

3. Conclusions

The differential equations were all solved on a Univac machine as initial-value problems with steps of 0·02 in η in the final stage. First and second differences of the functions and first four derivatives were calculated. The asymptotic form when compared with the numerical results gave the various constants. The functions behaved numerically as predicted by the analysis. This numerical verification settles the question of the construction of the solution up to the fourth term.

The solution obtained involves two undetermined constants A and B which will be determined by joining the above solution onto the leading edge solution. As far as the author is aware, no correct solution for this region has yet been found. In the external potential flow there are more than two undetermined b's although there are effectively only two independent undetermined constants related to A and B. At the $n = 2$ stage b_{10} is undetermined. At the $n = 4$ stage b_{11}, b_{21}, b_{22}, b_{32}, b_{33} are determined and b_{10}, b_{20} are undetermined. b_{31}, b_{30} are also unknown but are related to b_{10} and b_{20}. b_{31} depends only on b_{10} and b_{30} on b_{10} and b_{20}.

Thus, in conclusion, the functions $g_2(\eta)$, $h_3(\eta)$ (identically zero), $g_3(\eta)$ (identically zero), $f_3(\eta)$, $k_4(\eta)$ (identically zero) are all determined while $f_2(\eta)$, $g_4(\eta)$, $f_4(\eta)$ are undetermined. $f_2(\eta)$ and $g_4(\eta)$ depend on one undetermined constant A, introduced at the $n = 2$ stage, while $f_4(\eta)$ depends on two undetermined constants A and B, the second, B, being introduced at the $n = 4$ stage.

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† Lodged with the Editor.

Appendix

The general form of the solutions to equations (5) is discussed in §2 above and the particular cases $n = 2$, $n = 3$ are considered there. Although in principle the $n = 4$ case is not essentially different, it is given here for reference. Note that a further undetermined constant is introduced here.

Case $n = 4$. The equation for $k_4(\eta)$ is, from (5),

$$L_4(k_4) = 0,$$

where $k_4 \sim -b_{33}$. By the same argument as for h_3 and g_3 (see §2 above) $k_4 \equiv 0$ and $b_{33} = 0$. This fact is dependent on $b_4^{(2)}$ not being zero, which again was found to be the case numerically.

The function $h_4(\eta)$ satisfies

$$\begin{aligned} L_4(h_4) &= g_2 g_2'''' - 3g_2' g_2'' + 3(f_0' k_4'' - f_0'''' k_4) \\ &= g_2 g_2'''' - 3g_2' g_2'', \end{aligned} \tag{A 1}$$

since $k_4 \equiv 0$. Let $y_4^{(4)}$ be the particular integral of (A 1). Since the right-hand side asymptotes to zero,

$$y_4^{(4)} \sim a_4^{(4)} + b_4^{(4)} E_4 + c_4^{(4)} H_4.$$

From (4), $h_4 \sim -b_{32} + 2b_{22}\eta = -b_{32}$, since $b_{22} = 0$ from above. Thus, we can combine $y_4^{(2)}$ and $y_4^{(4)}$ to annul the E_4 term to give

$$h_4(\eta) = y_4^{(4)} - b_4^{(4)} y_4^{(2)} / b_4^{(2)}, \quad b_{32} = (b_4^{(4)} a_4^{(2)} - b_4^{(2)} a_4^{(4)}) / b_4^{(2)}, \tag{A 2}$$

where, since $b_4^{(2)} \neq 0$, it is necessary that $b_4^{(4)}$ is not equal to zero, which was found to be the case numerically.

Table 4† gives $h_4(\eta)$, $h_4'(\eta)$, table 5 $h_4''(0)$, and $5{\cdot}948 \leqslant b_{32} \leqslant 5{\cdot}949$.

The function $g_4(\eta)$ satisfies (see (5) and (6))

$$\begin{aligned} L_4(g_4) = {} & (-g_2 g_2'''' + g_2' g_2'' - 2g_2' f_0 - 10 g_2 f_0' - 12 g_2'') \\ & + 2\eta(2g_2''' + f_0 g_2'' - f_0'' g_2) \\ & + 2\eta^2 (f_0' g_2'' + f_0'' g_2') + 2(f_0' h_4'' - f_0'''' h_4) \\ & + (g_2 f_2''' - 3g_2' f_2'' - 3g_2'' f_2' + g_2''' f_2). \end{aligned} \tag{A 3}$$

The right-hand side of (A 3) contains a term multiplied by the unknown constant A and comes from the ${}_2f_2$ (see (12)) contribution in the last bracket. Let $y_4^{(5)}$ be the particular integral excluding the ${}_2f_2$ function and its derivatives and $y_4^{(6)}$ the particular integral with the ${}_2f_2$ function and its derivatives only and with $A = 1$. The right side of the equation for $y_4^{(5)}$ asymptotes to $-20b_{11}$ and so

$$y_4^{(5)} \sim -b_{11}\eta^2 + d_4^{(5)}\eta + a_4^{(5)} + b_4^{(5)} E_4 + c_4^{(5)} H_4.\ddagger$$

The right side of the equation for $y_4^{(6)}$ asymptotes to zero, so

$$y_4^{(6)} \sim a_4^{(6)} + b_4^{(6)} E_4 + c_4^{(6)} H_4.$$

Since (4) give

$$g_4(\eta) \sim (\tfrac{3}{4}\pi^2 b_{33} - b_{31}) + 2\eta(b_{21} - b_{22}) - b_{11}\eta^2 = -b_{31} + 2\eta b_{21} - b_{11}\eta^2,$$

† Lodged with the Editor.

‡ The d_4's are constants, none of which is numerically zero.

the appropriate linear combination of $y_4^{(2)}$, $y_4^{(3)}$, $y_4^{(5)}$ and $y_4^{(6)}$ to annul the E_4 term and give the correct asymptotic form above is

$$\left.\begin{aligned} g_4(\eta) &= (1/\alpha b_4^{(2)})\,[8(d_4^{(5)}-2b_{21})\,b_4^{(3)}-\alpha b_4^{(5)}]\,y_4^{(2)} \\ &\quad + (8/\alpha)\,(2b_{21}-d_4^{(5)})\,y_4^{(3)}+y_4^{(5)} \\ &\quad + (A/b_4^{(2)})\,(b_4^{(2)}y_4^{(6)}-b_4^{(6)}y_4^{(2)}), \\ g_4(\eta) &= {}_1g_4(\eta)+A\,{}_2g_4(\eta), \quad b_{31} = {}_1b_{31}+A\,{}_2b_{31}, \end{aligned}\right\} \tag{A 4}$$

where ${}_1g_4$, ${}_2g_4$ are defined by (A 4). In a similar way to the above, the existence of ${}_1g_4(\eta)$, ${}_2g_4(\eta)$ depends on all of $b_4^{(3)}$, $b_4^{(5)}$, $b_4^{(6)}$ ($b_4^{(2)}$ has already been shown to be non-zero) being numerically non-zero, which was found to be the case. Thus

$${}_1g_4(\eta) \sim -b_{11}\eta^2+2\eta b_{21}+{}_1b_{31}, \quad {}_2g_4(\eta) \sim {}_2b_{31},$$

$$\left.\begin{aligned} \text{where } {}_1b_{31} &= (a_4^{(2)}/\alpha b_4^{(2)})\,[8(2b_{21}-d_4^{(5)})+\alpha b_4^{(5)}]+(8a_4^{(3)}/\alpha)\,(d_4^{(5)}-2b_{21})-a_4^{(5)}, \\ {}_2b_{31} &= (b_4^{(2)}a_4^{(6)}-b_4^{(6)}a_4^{(2)})/b_4^{(2)}. \end{aligned}\right\} \tag{A 5}$$

${}_1g_4(\eta)$, ${}_2g_4(\eta)$ and their first derivatives are given in table 4,† ${}_1g_4''(0)$, ${}_2g_4''(0)$ in table 5, and $65{\cdot}794 \leqslant {}_1b_{31} \leqslant 65{\cdot}798$ and $9{\cdot}952 \leqslant {}_2b_{31} \leqslant 9{\cdot}953$.

$f_0''(0)$	${}_1f_2''(0)$	${}_2f_2''(0)$	$g_2''(0)$	$f_3''(0)$	$h_4''(0)$
1·3282	0·0000	1·3282	2·2058	14·6424	−4·0819
${}_1g_4''(0)$	${}_2g_4''(0)$	${}_1f_4''(0)$	${}_2f_4''(0)$	${}_3f_4''(0)$	${}_4f_4''(0)$
−69·3414	−8·1177	−515·6479	−97·7783	2·7547	21·0582

TABLE 5

Finally, $f_4(\eta)$ satisfies, from (5) and (6),

$$\begin{aligned} L_4(f_4) &= [3f_0g_2'+10g_2''+17f_0'g_2+2\eta f_0''g_2-2\eta^3f_0''(\eta f_0'-f_0)] \\ &\quad +[-(g_2f_2'''+12f_2''+2f_0f_2'-g_2''f_2'+10f_0'f_2) \\ &\quad +2\eta(2f_2'''+f_0f_2''-f_0''f_2) \\ &\quad +2\eta^2(f_0'f_2''+f_0''f_2')+(f_0'g_4''-f_0'''g_4)] \\ &\quad +[f_2f_2'''-3f_2'f_2'']. \end{aligned} \tag{A 6}$$

The unknown constant A appears on the right-hand side of (A 6), as does A^2. Let $y_4^{(7)}$, $y_4^{(8)}$, $y_4^{(9)}$ be the particular integrals with the right side of (A 6) the coefficients of A^0, A, A^2, respectively. The asymptotic form of the right of (A 6) is

$$(30b_{11}-20\,{}_1b_{10})-20\,{}_2b_{10}A.$$

Thus

$$\left.\begin{aligned} y_4^{(7)} &\sim (\tfrac{3}{2}b_{11}-{}_1b_{10})\,\eta^2+d_4^{(7)}\eta+a_4^{(7)}+b_4^{(7)}E_4+c_4^{(7)}H_4, \\ y_4^{(8)} &\sim -{}_2b_{10}\eta^2+d_4^{(8)}\eta+a_4^{(8)}+b_4^{(8)}E_4+c_4^{(8)}H_4, \\ y_4^{(9)} &\sim a_4^{(9)}+b_4^{(9)}E_4+c_4^{(9)}H_4. \end{aligned}\right\} \tag{A 7}$$

The $d_4^{(7)}$ and $d_4^{(8)}$ were found to be non-zero. From (4),

$$\begin{aligned} f_4(\eta) &\sim (\tfrac{1}{4}\pi^2b_{32}-b_{30})+\eta(2b_{20}-b_{21}-\tfrac{1}{2}\pi^2b_{22})+\eta^2(\tfrac{3}{2}b_{11}-b_{10}) \\ &= (\tfrac{1}{4}\pi^2b_{32}-b_{30})+\eta(2b_{20}-b_{21})+\eta^2(\tfrac{3}{2}b_{11}-b_{10}). \end{aligned}$$

† Lodged with the Editor.

$f_4(\eta)$ is given by a linear combination of the y_4's, namely

$$f_4(\eta) = By_4^{(2)} + Cy_4^{(3)} + y_4^{(7)} + Ay_4^{(8)} + A^2y_4^{(9)},$$

where B, C are at our disposal. Since b_{20} is as yet undetermined, C cannot be determined, so b_{30} is not determined. To annul the E_4 term

$$Bb_4^{(2)} + Cb_4^{(3)} + b_4^{(7)} + Ab_4^{(8)} + A^2b_4^{(9)} = 0,$$

and comparison with the asymptotic form for $f_4(\eta)$ above gives

$$\tfrac{1}{8}C\alpha + d_4^{(7)} + Ad_4^{(8)} = 2b_{20} - b_{21}.$$

Since, at this stage, A is a function of b_{10}, C is a function of b_{20}, b_{10} and so b_{30} is a function of b_{10}, b_{20}. Therefore there are *two* undetermined constants at this stage b_{10} and b_{20}. b_{30} is given in terms of b_{10} and b_{20}. Thus, only two of the constants are undetermined and available for joining onto the leading edge solution. We write

$$\left.\begin{aligned} f_4(\eta) &= {}_1f_4 + A\,{}_2f_4 + A^2{}_3f_4 + B\,{}_4f_4, \\ b_{30} &= {}_1b_{30} + A\,{}_2b_{30} + A^2{}_3b_{30} + B\,{}_4b_{30}, \end{aligned}\right\} \tag{A 8}$$

where B is an unknown constant which like A (or like two of b_{10}, b_{20}, b_{30}) is available to make a smooth join of this solution onto the leading edge one. The ${}_if_4(\eta)$ $(i = 1, 2, 3, 4)$, are

$$\left.\begin{aligned} {}_4f_4(\eta) &= (16/\alpha b_4^{(2)})\,(b_4^{(2)}y_4^{(3)} - b_4^{(3)}y_4^{(2)}) \sim 2\eta - {}_4b_{30}, \\ {}_3f_4(\eta) &= (1/b_4^{(2)})\,(b_4^{(2)}y_4^{(9)} - b_4^{(9)}y_4^{(2)}) \sim -{}_3b_{30}, \\ {}_2f_4(\eta) &= (1/b_4^{(2)})\,(b_4^{(2)}y_4^{(8)} - b_4^{(8)}y_4^{(2)}) - \tfrac{1}{2}d_4^{(8)}\,{}_4f_4 \sim -\beta\eta^2 - {}_2b_{30}, \\ {}_1f_4(\eta) &= y_4^{(7)} - (8/\alpha)\,(b_{21} + d_4^{(7)})\,y_4^{(3)} + (1/b_4^{(2)})\,[(8b_4^{(3)}/\alpha)\,(b_{21} + d_4^{(7)}) - b_4^{(7)}]\,y_4^{(2)} \\ &\sim (\tfrac{3}{2}b_{11} - {}_1b_{10})\,\eta^2 - b_{21}\eta + (\tfrac{1}{4}\pi^2 b_{32} - {}_1b_{30}). \end{aligned}\right| \tag{A 9}$$

Note that $b_{20} = B$. As is necessary and what was found numerically, $b_4^{(7)}$, $b_4^{(8)}$, $b_4^{(9)}$ are all non-zero. Note that a second undetermined constant is introduced at this stage. These functions and their first derivatives are given in tables 2, 3,† their second derivatives at $\eta = 0$ in table 5, and the constants

$$499{\cdot}523 \leqslant {}_1b_{30} \leqslant 499{\cdot}526, \quad 99{\cdot}561 \leqslant {}_2b_{30} \leqslant 99{\cdot}562,$$
$$-1{\cdot}533 \leqslant {}_3b_{30} \leqslant -1{\cdot}532, \quad \text{and} \quad {}_4b_{30} = 23{\cdot}903.$$

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† Lodged with the Editor.